INFORMATION AND CONTROL 20, 38–45 (1972)

A Directed-Divergence Function of Type β*

P. N. RATHIE AND PL. KANNAPPAN

Faculty of Mathematics, University of Waterloo, Waterloo, Ontario

A new concept of directed-divergence function of type β is introduced in this paper. This concept is used in obtaining a directed-divergence of type β which generalizes Kullback's directed-divergence and has a relation with Rényi's information-gain of order β. This relation can be used to give another characterization of information-gain of order β. A characterization theorem for the directed-divergence of type β is proved with the help of a functional equation.

1. INTRODUCTION

Let $P = (p_1, p_2, ..., p_n)$, $p_i \geqslant 0$ with $\sum_{i=1}^n p_i = 1$ and $Q = (q_1, q_2, ..., q_n)$, $q_i \geqslant 0$ with $\sum_{i=1}^n q_i = 1$ be two finite discrete probability distributions. Then Kullback's directed-divergence (Kullback, 1959) for the two probability distributions P and Q is defined as,

$$I_n^1 \begin{pmatrix} p_1, ..., p_n \\ q_1, ..., q_n \end{pmatrix} = \sum_{i=1}^n p_i \log_2(p_i q_i^{-1}). \tag{1.1}$$

Also, Rényi's information-gain of order β (Rényi, 1961) is given by

$$I^\beta(P \mid Q) = (\beta - 1)^{-1} \log_2 \left(\sum_{i=1}^n p_i^\beta q_i^{1-\beta} \right), \qquad \text{for} \quad \beta \neq 1. \tag{1.2}$$

In this paper, we introduce another generalization of (1.1) called the directed-divergence of type β:

$$I_n^\beta \begin{pmatrix} p_1, ..., p_n \\ q_1, ..., q_n \end{pmatrix} = \left(\sum_{i=1}^n p_i^\beta q_i^{1-\beta} - 1 \right) (2^{\beta-1} - 1)^{-1}, \qquad \text{for} \quad \beta \neq 1, \tag{1.3}$$

which reduces to (1.1) when $\beta \to 1$. The quantity I_n^β is characterized by a set

* This work is partially supported by a Department of University Affairs (Ontario, Canada) grant.

of three postulates in the next section. In the process of characterization of I_n^β a functional equation is obtained which may be regarded as the fundamental functional equation defining a new directed-divergence function of type β for (1.3).

Clearly from (1.2) and (1.3), we get the following relation involving the directed-invergence of type β and the information-gain of order β:

$$I_n^\beta \begin{pmatrix} p_1, ..., p_n \\ q_1, ..., q_n \end{pmatrix} = (2^{\beta-1} - 1)^{-1}(2^{(\beta-1)I_\beta(P|Q)} - 1), \qquad \text{for} \quad \beta \neq 1. \quad (1.4)$$

The results of this paper also give new characterization of Rényi's information of type β by means of a functional equation.

Very recently, (1.1) was characterized (Campbell, 1970) for arbitrary probability space by a set of six postulates. Rathie (1970) has recently characterized (1.3) by means of a functional inequality for $n > 2$.

The case $\beta = 1$ giving new characterization for (1.1) by means of a functional equation will be discussed by us elsewhere.

2. Characterization Theorem

We prove the following theorem regarding the directed-divergence of type β, that is (1.3).

THEOREM. *If K_n, $n = 2, 3,...$ satisfies the following relations,*

(i) $$K_n \begin{pmatrix} p_1, ..., p_n \\ q_1, ..., q_n \end{pmatrix}$$
$$= K_{n-1} \begin{pmatrix} p_1 + p_2, p_3, ..., p_n \\ q_1 + q_2, q_3, ..., q_n \end{pmatrix}$$
$$+ (p_1 + p_2)^\beta (q_1 + q_2)^{1-\beta} K_2 \begin{pmatrix} p_1/(p_1 + p_2), p_2/(p_1 + p_2) \\ q_1/(q_1 + q_2), q_2/(q_1 + q_2) \end{pmatrix},$$

with $p_1 + p_2, q_1 + q_2 > 0$;

(ii) $K_3 \begin{pmatrix} p_1, p_2, p_3 \\ q_1, q_2, q_3 \end{pmatrix}$ *is a symmetric function of its variables* $\begin{Bmatrix} p_i \\ q_i \end{Bmatrix}$, $i = 1, 2, 3$;

(iii) $K_2 \begin{pmatrix} 1, & 0 \\ \frac{1}{2}, & \frac{1}{2} \end{pmatrix} = 1,$

then

$$K_n \equiv I_n^\beta = \left(\sum_{i=1}^{n} p_i^\beta q_i^{1-\beta} - 1\right)(2^{\beta-1} - 1)^{-1}, \qquad \text{for} \quad \beta \neq 1. \tag{1.3}$$

Proof. We define

$$f(x, y) = K_2\begin{pmatrix} x, 1 - x \\ y, 1 - y \end{pmatrix}, \qquad x, y \in [0, 1]. \tag{2.1}$$

First we will prove the following lemma giving rise to a functional equation related to f given by (2.1).

LEMMA 1. *The function $f : I \times I \to R$ given by (2.1) where $I = [0, 1]$ and R, the real numbers, satisfies the functional equation*

$$f(x, y) + (1 - x)^\beta (1 - y)^{1-\beta} f\left(\frac{u}{1 - x}, \frac{v}{1 - y}\right)$$

$$= f(u, v) + (1 - u)^\beta (1 - v)^{1-\beta} f\left(\frac{x}{1 - u}, \frac{y}{1 - v}\right),$$

for all $x, y, u, v \in [0, 1)$ with $x + u$, $y + v \in I$ and the boundary conditions

$$f(0, 0) = f(1, 1) = 0 \tag{2.3}$$

and

$$f(0, \tfrac{1}{2}) = f(1, \tfrac{1}{2}) = 1. \tag{2.4}$$

Proof. Taking $n = 3$ in (i), (i) becomes

$$K_3\begin{pmatrix} p_1, p_2, p_3 \\ q_1, q_2, q_3 \end{pmatrix}$$

$$= K_2\begin{pmatrix} p_1 + p_2, p_3 \\ q_1 + q_2, q_3 \end{pmatrix} + (p_1 + p_2)^\beta (q_1 + q_2)^{1-\beta} K_2 \begin{pmatrix} \dfrac{p_1}{p_1 + p_2}, \dfrac{p_2}{p_1 + p_2} \\ \dfrac{q_1}{q_1 + q_2}, \dfrac{q_2}{q_1 + q_2} \end{pmatrix} \tag{2.5}$$

for $p_1 + p_2$, $q_1 + q_2 > 0$.

Interchanging p_1 and p_2 and q_1 and q_2 in (2.5) and using (ii), (2.5) gives

$$K_2\begin{pmatrix} \frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2} \\ \frac{q_1}{q_1+q_2}, \frac{q_2}{q_1+q_2} \end{pmatrix} = K_2\begin{pmatrix} \frac{p_2}{p_1+p_2}, \frac{p_1}{p_1+p_2} \\ \frac{q_2}{q_1+q_2}, \frac{q_1}{q_1+q_2} \end{pmatrix}, \tag{2.6}$$

for $p_1, p_2, q_1, q_2 \in I$ with $p_1 + p_2, q_1 + q_2 \in (0, 1]$.

Hence (2.1) and (2.6) yield

$$f\left(\frac{p_1}{p_1+p_2}, \frac{q_1}{q_1+q_2}\right) = f\left(\frac{p_2}{p_1+p_2}, \frac{q_2}{q_1+q_2}\right)$$

which is equivalent to

$$f(x, y) = f(1-x, 1-y) \qquad \text{for all} \quad x, y \in I. \tag{2.7}$$

Rewriting (ii), we have

$$K_3\begin{pmatrix} x, 1-x-u, u \\ y, 1-y-v, v \end{pmatrix} = K_3\begin{pmatrix} u, 1-x-u, x \\ v, 1-y-v, y \end{pmatrix}, \tag{2.8}$$

where $x, y, u, v, 1-x-u, 1-y-v \in I$.

Now (2.8) with the help of (i) for $n = 3$ and (2.1) gives

$$f(1-u, 1-v) + (1-u)^\beta(1-v)^{1-\beta} f\left(\frac{x}{1-u}, \frac{y}{1-v}\right)$$

$$= f(1-x, 1-y) + (1-x)^\beta(1-y)^{1-\beta} f\left(\frac{u}{1-x}, \frac{v}{1-y}\right), \tag{2.9}$$

for $x, y, u, v \in [0, 1)$ with $x + u, y + v \in I$.

In view of (2.7), (2.9) takes the form

$$f(u, v) + (1-u)^\beta(1-v)^{1-\beta} f\left(\frac{x}{1-u}, \frac{y}{1-v}\right)$$

$$= f(x, y) + (1-x)^\beta(1-y)^{1-\beta} f\left(\frac{u}{1-x}, \frac{v}{1-y}\right),$$

which is precisely (2.2).

Also (ii) on using (i) gives

$$K_3\begin{pmatrix} 1, 0, 0 \\ 1, 0, 0 \end{pmatrix} = 2f(1, 1),$$

and

$$K_3\begin{pmatrix}0, 1, 0\\0, 1, 0\end{pmatrix} = f(1, 1) + f(0, 0),$$

giving,

$$f(1, 1) = f(0, 0). \tag{2.10}$$

Putting $x = y = \frac{1}{2}$, $u = v = 0$ in (2.2), we have,

$$f(1, 1) = \tfrac{1}{2}f(0, 0). \tag{2.11}$$

Thus, from (2.10) and (2.11), we get

$$f(1, 1) = f(0, 0) = 0,$$

which is precisely (2.3).

Furthermore from (iii), (2.1) and (2.7) for $x = 0$ and $y = \frac{1}{2}$, result

$$f(1, \tfrac{1}{2}) = f(0, \tfrac{1}{2}) = 1,$$

which is precisely (2.4).

Thus Lemma 1 is proved.

Definition 1. A function $f: I \times I \to R$ satisfying the functional equation (2.2) and the boundary conditions $f(0, 0) = f(1, 1)$ and (2.4) is called a directed-divergence function of type β.

Now we will describe all the solutions of the functional equation (2.2) under the boundary conditions (2.3) and (2.4).

Lemma 2. *Every solution f of* (2.2) *satisfying the additional conditions* (2.3) *and* (2.4) *is given by*

$$f(x, y) = [x^\beta y^{1-\beta} + (1 - x)^\beta(1 - y)^{1-\beta} - 1](2^{\beta-1} - 1)^{-1}, \tag{2.12}$$

for $x, y \in I$, in which the notation $0^\alpha = 0$ $(\alpha \neq 0)$ is used.

Remark 1. In order to obtain the solution of (1.3) in the desired form (2.12) for all $x, y \in I$, the convention $0^\alpha = 0$ $(\alpha \neq 0)$ is used, but nowhere in the proof of the theorem, $0^\alpha = 0$ is used.

Proof. Substitution of $p = u/(1-x)$, $q = v/(1-y)$, $r = 1-x$ and $s = 1-y$ in (2.2), reduce (2.2) to

$$f(1-r, 1-s) + r^\beta s^{1-\beta} f(p, q) = f(pr, qs) + (1-pr)^\beta (1-qs)^{1-\beta} f\left(\frac{1-r}{1-pr}, \frac{1-s}{1-qs}\right), \quad (2.13)$$

for all $p, q \in I$, $r, s \in (0, 1]$ such that $pr \neq 1$ and $qs \neq 1$.

Let $r = 1$, $s = \frac{1}{2}$ in (2.13), then we have

$$f(0, \tfrac{1}{2}) + 2^{\beta-1} f(p, q) = f(p, q/2) + (1-p)^\beta (1-q/2)^{1-\beta} f\left(0, \frac{1}{2-q}\right), \quad (2.14)$$

for $p \in [0, 1)$, $q \in [0, 1]$.

Taking $q = 0$ in (2.14) and using (2.4) we get

$$f(p, 0) = [(1-p)^\beta - 1](2^{\beta-1} - 1)^{-1}, \qquad \text{for} \quad p \in [0, 1). \quad (2.15)$$

Putting $p = 1$, $q = 0$ in (2.13) we get

$$f(1-r, 1-s) + r^\beta s^{1-\beta} f(1, 0) = f(r, 0) + (1-r)^\beta f(1, 1-s), \quad \text{for} \quad r \in (0, 1), \ s \in (0, 1]. \quad (2.16)$$

Now (2.15) and (2.16) for $s = 1$ give

$$f(1, 0) = (1 - 2^{\beta-1})^{-1}. \quad (2.17)$$

Again taking $r = 1$, $q = 0$ in (2.13), we get

$$f(0, 1-s) + s^{1-\beta} f(p, 0) = f(p, 0) + (1-p)^\beta f(0, 1-s), \quad (2.18)$$

which on using (2.15) gives us

$$f(0, 1-s) = (s^{1-\beta} - 1)(2^{\beta-1} - 1)^{-1}, \qquad \text{for} \quad s \in (0, 1]. \quad (2.19)$$

Also for $p = 0$, $q = 1$, (2.13) becomes

$$f(1-r, 1-s) + r^\beta s^{1-\beta} f(0, 1) = f(0, s) + (1-s)^{1-\beta} f(1-r, 1), \quad \text{for} \quad r \in (0, 1], \ s \in (0, 1). \quad (2.20)$$

Now (2.20) with $r = 1$ and (2.19) give

$$f(0, 1) = (1 - 2^{\beta-1})^{-1}. \quad (2.21)$$

Further (2.16) and (2.20) on using (2.17) and (2.21) give

$$f(r,0)+(1-r)^{\beta}f(1,1-s)=f(0,s)+(1-s)^{1-\beta}f(1-r,1), \quad \text{for} \quad r,s\in(0,1). \tag{2.22}$$

From (2.22) for $s=\frac{1}{2}$, (2.15) and (2.4), we have

$$f(1-r,1)=[(1-r)^{\beta}-1](2^{\beta-1}-1)^{-1}, \qquad \text{for} \quad r\in(0,1). \tag{2.23}$$

Hence (2.22), (2.23), (2.15) and (2.19) give

$$f(1,1-s)=[(1-s)^{1-\beta}-1](2^{\beta-1}-1)^{-1}, \qquad \text{for} \quad s\in(0,1). \tag{2.24}$$

Thus (2.15), (2.17), (2.19), (2.21), (2.23) and (2.24) give $f(x,0)$, $f(0,x)$, $f(1,x)$, $f(x,1)$ for $x\in[0,1]$.

Also (2.20), (2.21), (2.19) and (2.23) give

$$f(1-r,1-s)=[r^{\beta}s^{1-\beta}+(1-r)^{\beta}(1-s)^{1-\beta}-1](2^{\beta-1}-1)^{-1}, \quad \text{for} \quad r,s\in(0,1). \tag{2.25}$$

It is easy to see from (2.15), (2.17), (2.19), (2.21), (2.23), (2.24) and (2.25) that f is given by (2.12) with the notation $0^{\alpha}=0$ ($\alpha\neq 0$) and that (2.7) is true for all $x,y\in[0,1]$. This completes the proof of Lemma 2.

Now, we return to the proof of the theorem. By successive applications of (i) and the use of (2.1) give

$$K_n=I_n^{\beta}\begin{pmatrix}p_1,\ldots,p_n\\ q_1,\ldots,q_n\end{pmatrix}=\sum_{i=2}^{n} r_i^{\beta}s_i^{1-\beta}f\left(\frac{p_i}{r_i},\frac{q_i}{s_i}\right), \tag{2.26}$$

where $r_i=p_1+\cdots+p_i$, $s_i=q_1+\cdots+q_i$ for $i=1,2,\ldots,n$ with $r_n=s_n=1$.

Thus (2.26) and (2.12) prove the theorem.

Definition 2. If f is a directed-divergence function of type β ($\beta\neq 1$) as given by Definition 1, then I_n^{β} given by (2.26) [also (1.3)] is called the directed-divergence of type β ($\beta\neq 1$).

Several properties like null-information, non-negativity, symmetry, expansibility, etc., can be easily derived for I_n^β. We mention below a strong non-additivity property for I_n^β.

$$I_{mn}^\beta \begin{pmatrix} p_1p_{11}, p_1p_{21}, \dots, p_1p_{m1}, \dots & \dots, p_np_{1n}, p_np_{2n}, \dots, p_np_{mn} \\ q_1q_{11}, q_1q_{21}, \dots, q_1q_{m1}, \dots & \dots, q_nq_{1n}, q_nq_{2n}, \dots, q_nq_{mn} \end{pmatrix}$$

$$= I_n^\beta \begin{pmatrix} p_1, \dots, p_n \\ q_1, \dots, q_n \end{pmatrix} + \sum_{i=1}^n p_i^\beta q_i^{1-\beta} I_m^\beta \begin{pmatrix} p_{1i}, \dots, p_{mi} \\ q_{1i}, \dots, q_{mi} \end{pmatrix}, \tag{2.27}$$

where $\sum_{j=1}^m p_{ji} = 1$ and $\sum_{j=1}^m q_{ji} = 1$ for all $i = 1, \dots, n$.

An interesting special case of (2.27) is given below.

$$I_{mn}^\beta \begin{pmatrix} p_1P_1, p_1P_2, \dots, p_1P_m, \dots & \dots, p_nP_1, p_nP_2, \dots, p_nP_m \\ q_1Q_1, q_1Q_2, \dots, q_1Q_m, \dots & \dots, q_nQ_1, q_nQ_2, \dots, q_nQ_m \end{pmatrix}$$

$$= I_n^\beta \begin{pmatrix} p_1, \dots, p_n \\ q_1, \dots, q_n \end{pmatrix} + I_m^\beta \begin{pmatrix} P_1, \dots, P_m \\ Q_1, \dots, Q_m \end{pmatrix}$$

$$+ (2^{\beta-1} - 1) I_n^\beta \begin{pmatrix} p_1, \dots, p_n \\ q_1, \dots, q_n \end{pmatrix} I_m^\beta \begin{pmatrix} P_1, \dots, P_m \\ Q_1, \dots, Q_m \end{pmatrix}, \tag{2.28}$$

where $\sum_{j=1}^m P_j = 1$ and $\sum_{j=1}^m Q_j = 1$.

RECEIVED: January 20, 1971; REVISED: August 4, 1971

REFERENCES

CAMPBELL, L. L. (1970), Characterization of entropy in arbitrary probability spaces, pp. 1–26. Queen's University Preprint No. 32, Kingston.

KULLBACK, S. (1959), "Information Theory and Statistics," John Wiley and Sons, New York.

RATHIE, P. N. (1970), Some characterization theorems for generalized measures of uncertainty and information, *Meeting on Information Measures*, University of Waterloo, Waterloo, April 10–14.

RÉNYI, A. (1961), On measures of entropy and information, *Proc. Fourth Berkeley Symposium Math. Statist. and Probability* 1, 547–561.